Realization of Lévy walks as Markovian stochastic processes

Ihor Lubashevsky,[1,2,3] Rudolf Friedrich,[4,5] and Andreas Heuer[2,5]
[1]A.M. Prokhorov General Physics Institute, Russian Academy of Sciences, Vavilov Strasse 38, 119991 Moscow, Russia
[2]Institut für Physikalische Chemie, Westfälische Wilhelms Universität Münster, Corrensstrasse 30, 48149 Münster, Germany
[3]Moscow Technical University of Radioengineering, Electronics, and Automation, Vernadsky 78, 119454, Moscow, Russia
[4]Institut für Theoretische Physik, Westfälische Wilhelms Universität Münster, Wilhelm-Klemm. 9, 48149 Münster, Germany
[5]Center of Nonlinear Science CeNoS, Westfälische Wilhelms Universität Münster, 48149 Münster, Germany
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Based on multivariate Langevin processes we present a realization of Lévy flights as a continuous process. For the simple case of a particle moving under the influence of friction and a velocity-dependent stochastic force we explicitly derive the generalized Langevin equation and the corresponding generalized Fokker-Planck equation describing Lévy flights. Our procedure is similar to the treatment of the Kramers-Fokker-Planck equation in the Smoluchowski limit. The proposed approach may open a way to treat Lévy flights in inhomogeneous media or systems with boundaries in the future.

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It has become evident that Brownian random walks may be a too simple description of diffusion processes in complex systems such as the motion of tracer particles in turbulent flows [1], the diffusion of particles in random media [2], human travel behavior and spreading of epidemics [3], or economic time series in finance [4]. A variable x corresponding to such a process can frequently exhibit the dynamics described by the notion of superfast diffusion, where the characteristic value $\bar{x}$ of the variable x demonstrates scaling behavior $[\bar{x}(t)]^2 \propto t^{2/\alpha}$ with $\alpha < 2$ (see, e.g., Ref. [5]). The meaning of $\bar{x}(t)$ can be explained by referring to the behavior of the distribution function $P(x)$ of the random variable x. The value $\bar{x}(t)$ divides the spatial scales into two groups, $|x| \ll \bar{x}(t)$ and $|x| \gg \bar{x}(t)$, that are characterized by different asymptotic behavior of the function $P(x)$. Roughly speaking, the distribution function $P(x)$ should depend actually on the ratio $x/\bar{x}(t)$, e.g., $P(x) = [\bar{x}(t)]^{-1} p[x/\bar{x}(t)]$ in a one-dimensional (1D) case. Also it should be noted that the notion of the superfast diffusion could be too simple to describe all the anomalous properties observed in the aforementioned systems. Nevertheless, the superdiffusion is worthy of individual consideration because its anomalous time dependence stems from the violation of spatial localization of Markovian random walks on small time scales, which is one of the fundamental properties forming the base of classical theory of random processes.

Brownian motion is described on the basis of Langevin equations or, in a statistical sense, by the Fokker-Planck equation (cf. [6]). A straightforward way to deal with anomalous diffusion is based on a generalization of the Langevin equations by replacing Gaussian white noise with Lévy noise [7]. Recently, there has been a great deal of research about superfast diffusion. It includes, in particular, a rather general analysis of the Langevin equation with Lévy noise (see, e.g., Ref. [8]) and the form of the corresponding Fokker-Planck equations [9,10], description of anomalous diffusion with power-law distributions of spatial and temporal steps [7,11], Lévy flights in heterogeneous media [12–14] and in external fields [15,16], first passage time analysis and escaping problem for Lévy flights [17–22], as well as processing experimental data for detecting the Lévy type behavior [23]. Besides, it should be noted that the attempt to consider Lévy flights in bounded systems (see, e.g., Refs. [24,25] and review [26] as well) has introduced the notion of Lévy walks being a non-Markovian process because of the necessity to bound the walker velocity. The problems arise because for a Lévy walk the second moment $\langle (x_t - x_i)^2 \rangle$ (x_i and x_t are the initial and final point, respectively) diverges.

The key point in constructing the mutually related pair of the stochastic Langevin equation and the nonlocal Fokker-Planck equation for superdiffusion is the Lévy-Gnedenko central limit theorem. It specifies the possible step distributions $P(\Delta x)$ which display universal behavior. In particular, for a symmetrical homogeneous 1D system superdiffusion can be regarded as a chain of steps $\{\Delta x\}$ of duration δt whose distribution function $P(\Delta x)$ exhibits the following asymptotic behavior for $|\Delta x| \gg \bar{x}(\delta t)$:

$$P(\Delta x) \sim \frac{[\bar{x}(\delta t)]^{\alpha}}{|\Delta x|^{\alpha+1}}. \tag{1a}$$

Naturally, for $|\Delta x| \gg \bar{x}(\delta t)$

$$P(\Delta x) \sim \frac{1}{\bar{x}(\delta t)}. \tag{1b}$$

The purpose of the present paper is to introduce a model which generates continuous Markovian trajectories, following the Lévy statistics, by using simple Gaussian but multiplicative noise for the time evolution of the velocity. The spatial dynamics naturally follows from this. A first step in this direction can be found in Refs. [30,31]. The differences to that work will be indicated below. For a fixed time scale δt we can recover the standard behavior of Lévy flights. However, we have full locality in the sense that a trajectory can be determined with any desired resolution. In other words, we propose a microscopic implementation of Lévy processes characterized by an arbitrary small time scale τ that can be chosen beforehand. When running time exceeds essentially this microscopic time scale, $t \gg \tau$, the corresponding random walks, as should be, are described by distribution (1a).

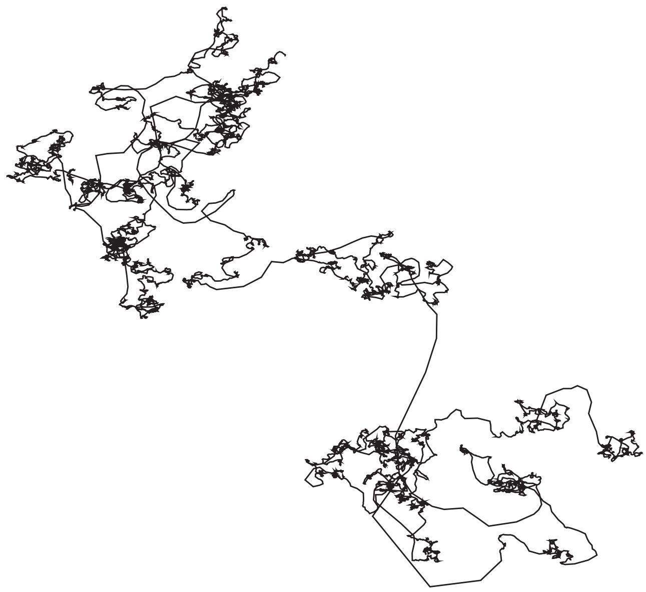

FIG. 1. Characteristic form of random walks described by the 2D analogy of model (2). The used system parameters correspond to the Lévy exponent $\alpha = 1.6$.

The specific 1D model under consideration, which can be easily generalized to arbitrary dimension, reads

$$\frac{dx}{dt} = v, \tag{2a}$$

$$\frac{dv}{dt} = -\frac{(\alpha+1)}{2\tau}v + \frac{1}{\sqrt{\tau}}g(v) * \xi(t). \tag{2b}$$

Here x and v are the position and the velocity of the walker, respectively, τ is the microscopic time scale mentioned above, the intensity of the Langevin random force is given by the function

$$g(v) = \sqrt{v_a^2 + v^2}, \tag{3}$$

with the parameter v_a measuring the intensity of the additive component of Langevin force, $\xi(t)$ is white noise such that $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$, and the parameter $\alpha \in (1,2)$. The Langevin equation (2b) is written in the Hänggi-Klimontovich form [27], which is indicated by the asterisk. The dynamics, resulting from a two-dimensional (2D) version of these equations, is visualized in Fig. 1.

The proposed model actually describes a certain stochastic self-acceleration of the particle motion in the $|v|$ space caused by nonlinearity of the Langevin random forces reflected in the dependence $g(v) \propto |v|$ for $|v| \gg v_a$. From the physical point the central question which has to be addressed is the origin of the multiplicative noise. In general multiplicative noise in nonlinear systems far from equilibrium arise quite naturally by projecting on physically relevant variables eliminating fast relaxing degrees of freedom (e.g., [6]).

The main purpose of our paper is to initiate the development of an approach to describing Lévy flights and Lévy walks using the *notion of continuous Markovian trajectories*. In this way it could become possible to construct a mathematical description of Lévy type processes in media with boundaries or spatially dependent kinetic coefficients. Naturally, transport phenomena in systems with a fractal advection field can be analyzed in the frameworks of the proposed model only at a phenomenological level.

The corresponding forward Fokker-Planck equation for the distribution function $\mathcal{P}(x-x_0, v, v_0, t)$ reads

$$\frac{\partial \mathcal{P}}{\partial t} = \frac{1}{2\tau}\frac{\partial}{\partial v}\left[g^2(v)\frac{\partial \mathcal{P}}{\partial v} + (\alpha+1)v\mathcal{P}\right] - \frac{\partial}{\partial x}[v\mathcal{P}], \tag{4}$$

where the values x_0 and v_0 specify the initial position of the walker. The distribution of the walker velocities v is determined by the partial distribution function

$$P_v(v, v_0, t) = \int_{-\infty}^{+\infty} \mathcal{P}(x-x_0, v, v_0, t)dx. \tag{5}$$

Using the corresponding Fokker-Planck equation for function (5) following directly from Eq. (4) we get

$$\langle v(t)\rangle = v_0 \exp\left[-\frac{(\alpha-1)}{2}\frac{t}{\tau}\right], \tag{6a}$$

$$\langle v^2(t)\rangle = v_0^2 \exp\left[(2-\alpha)\frac{t}{\tau}\right] \quad \text{for } v_0 \gtrsim v_a. \tag{6b}$$

These expressions characterize actually the relaxation of the velocity distribution (5) to its steady state form. Leaping ahead we can declare that model (2) does lead to the Lévy statistics so in what follows the analyzed random process will be referred to as random motion of Lévy walker.

The exponential decay of the first velocity moment (6a) demonstrates the fact that the Lévy walker "remembers" its initial velocity practically on time scales not exceeding the value τ. So, in particular, the stationary velocity distribution $P_v^{\mathrm{st}}(v)$ must be a symmetric function of v and, by virtue of Eq. (4), meets the equality

$$g^2(v)\frac{\partial P_v^{\mathrm{st}}}{\partial v} + (\alpha+1)vP_v^{\mathrm{st}} = 0. \tag{7}$$

Hence we immediately get the expression

$$P_v^{\mathrm{st}}(v) = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\alpha}{2}\right)}\frac{v_a^{\alpha}}{[g(v)]^{\alpha+1}}, \tag{8}$$

where $\Gamma(\cdots)$ is the Gamma function.

The exponential decay as well as divergence of the first and second moments of the particle velocity, respectively, expressions (6), indicate that the system relaxes to the stationary distribution (8) on time scales $t \gg \tau$. So, in some sense, the spatial steps of duration about τ are mutually independent. In other words, the value τ separates the time scales into two groups. On scales less than τ the particle motion is strongly correlated and has to be considered using both the phase variables x and v. Thus on a time scale $\delta t \gg \tau$ the particle displacements are mutually independent and the succeeding steps of the Lévy walker form a Markovian

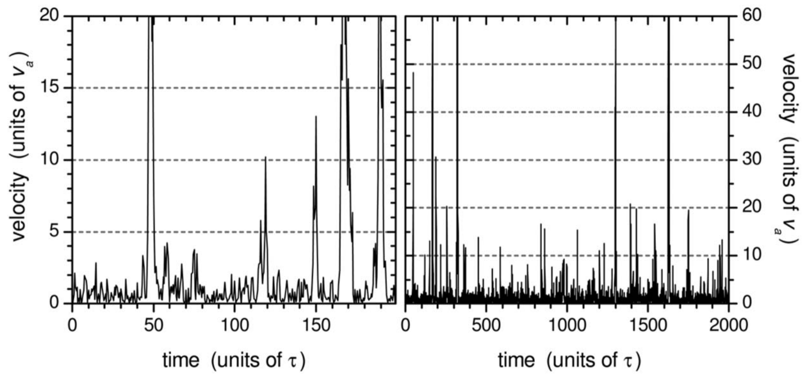

FIG. 2. Characteristic form of the time pattern $v(t)$ exhibited by the stochastic system (2). The individual windows depict the patterns on various scales. In the simulation the parameter $\alpha=1.6$ was used.

chain, with the particle velocity playing the role of Lévy noise. This scenario is exemplified in Fig. 2 for some realization of $v(t)$ following from Eq. (2b). Lévy flight events, i.e., the long-distance jumps of the Lévy walker, are due to large spikes of the time pattern $v(t)$ whose duration is about several τ. More precisely, the long-distance displacement Δx of a walker during a certain time interval δt is mainly caused by the velocity spike of maximal amplitude ϑ attained during the given interval, i.e., $\Delta x \sim \vartheta\tau$. For $\delta t \gg \tau$ the quantities $\{\vartheta\}$ are statistically independent of one another.

To analyze the properties of the velocity extrema $\{\vartheta\}$ we make use of the relation between the extremum statistics of Markovian processes and the first passage time distribution [28]. Namely, the probability function $\Phi(\vartheta,v,t)$ of the random variable ϑ and the probability $F(\vartheta,v,t)$ of passing the boundary $v=\pm\vartheta$ for the first time at moment t are related as (cf. formulas (7)-(9) in Ref. [28])

$$\Phi(\vartheta,v,t) = -\frac{\partial}{\partial\vartheta}\int_0^t F(\vartheta,v,t')dt'. \tag{9}$$

Here v is the initial velocity of the Lévy walker. Let us confine our consideration to a qualitative analysis of the deep tails of the extremum distribution. This case implies the inequality $\vartheta \gg \bar{\vartheta}(t)$, where $\bar{\vartheta}(t)$ is the typical value taken by the random variable ϑ during time t. By virtue of Eq. (6) the system loses its memory of the initial conditions within a time interval about τ. As a result the particle motion during time $t\gg\tau$ can be treated as a sequence of mutually independent steps of duration τ within which the system dynamics is strongly correlated. Then under the adopted assumptions the probability of the walker leaving the domain $(-\vartheta,\vartheta)$ ($\vartheta>0$) during time $t\gg\tau$ can be estimated as

$$\int_0^t F(\vartheta,v,t')dt' \sim \left\{1-\left[\int_{-\vartheta}^{+\vartheta} P_v^{\mathrm{st}}(u)du\right]^a\right\} \sim 2a\int_\vartheta^\infty P_v^{\mathrm{st}}(u)du \ll 1 \tag{10}$$

with the exponent $a\sim t/\tau$, provided the right-hand side of Eq. (10) is much less than unity. The integral over velocity entering expression (10) is no more than the probability of the walker crossing the boundary ϑ during one such step. Expressions (9) and (10) immediately give us the required asymptotics of the extremum distribution

$$\Phi(\vartheta,t) \sim \frac{t}{\tau}P_v^{\mathrm{st}}(\vartheta) \sim \frac{\bar{\vartheta}^\alpha(t)}{|\vartheta|^{\alpha+1}}, \tag{11}$$

where the quantity $\bar{\vartheta}(t)$ is given by the estimate $\bar{\vartheta}(t) \sim v_a(t/\tau)^{1/\alpha}$. The structure of asymptotics (11) shows us that the quantity $\bar{\vartheta}(t)$ is actually the scale introduced previously to characterize variations in the velocity extrema $\{\vartheta\}$.

If the spikes shown in Fig. 2 had the same shape than the normalized walker displacement $\Delta x/\vartheta$ would be a constant of the order of τ and the distribution function of the walker displacement $\Delta x = x - x_0$,

$$P_x(\Delta x, v_0, t) = \int_{-\infty}^{+\infty}\mathcal{P}(x-x_0,v,v_0,t)dv, \tag{12}$$

would be directly related to the distribution $\Phi(\vartheta,t)$ and, by virtue of Eq. (11), have form (1a) for $t\gg\tau$. Via numerical simulation we have determined the distribution of $\Delta x/\vartheta$ for given velocity extremum ϑ. The first and second moments of this distribution are shown in Fig. 3. As expected the average value of $\Delta x/\vartheta$ indeed approaches a constant c_τ (for $\alpha=1.6$ the value $c_\tau\approx1.6\tau$). However, the finite variance shows that the velocity spikes have some distribution in their shape. Thus, *a priori*, the distributions $\Phi(\vartheta)$ and $P_x(\Delta x)$ are not identical within the replacement $\Delta x \leftrightarrow c_\tau\vartheta\tau$. However, since the distribution of $\Delta x/\vartheta$ for fixed ϑ does *not* depend on ϑ (for large ϑ) one can directly write

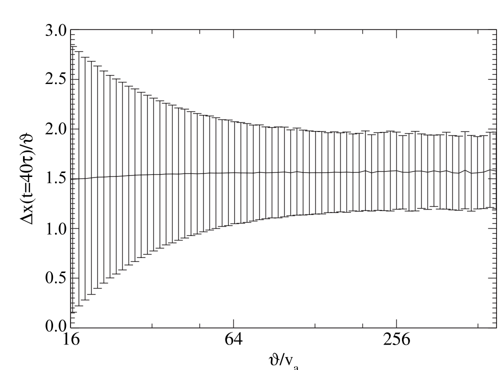

FIG. 3. The ratio $\Delta x/\vartheta$ for individual steps vs the values of the random variable ϑ. In simulation $\alpha=1.6$ was used.

$$P_x(\Delta x) \propto \int d\epsilon d\vartheta q(\epsilon)\vartheta^{-(1+\alpha)}\delta(\epsilon\vartheta + c_\tau\vartheta - \Delta x)$$
$$\propto \int d\epsilon q(\epsilon)[(\epsilon + c_\tau)/\Delta x]^{(1+\alpha)} \propto \Delta x^{-(1+\alpha)}, \quad (13)$$

where $q(\epsilon)$ is the distribution of the random variable $\epsilon := \Delta x/\vartheta - c_\tau$. Thus despite the variance in peak shapes the algebraic distribution of ϑ directly translates into an identical distribution for Δx, giving us the Lévy asymptotics of the walker displacements Δx during the time interval t,

$$P_x(\Delta x, t) \sim \frac{[\bar{x}(t)]^\alpha}{(\Delta x)^{\alpha+1}}, \quad (14)$$

where the characteristic time scale $\bar{x}(\delta t)$ of the walker displacement is estimated by the expression

$$\bar{x}(t) \sim v_a \tau^{(\alpha-1)/\alpha} t^{1/\alpha}. \quad (15)$$

The results obtained in the present paper can be strictly justified [29], leading in particular to the expression

$$c_\tau = \left[\frac{2\sin\left(\frac{\pi\alpha}{2}\right)\Gamma\left(\frac{2-\alpha}{2}\right)}{\sqrt{\pi}\alpha\Gamma\left(\frac{\alpha+1}{2}\right)}\right]^{1/\alpha}, \quad (16)$$

which gives the value $c_\tau \approx 1.6$ for $\alpha = 1.6$ in agreement with the simulation data.

A related model, using multiplicative noise for the velocity increment, has been suggested in Ref. [30]. In that work the displacement after time t is estimated from the sum of t/τ statistically independent velocity contributions. In the present work, a more careful analysis is performed where, as a key element, the maximum velocity in a given time interval enters. Furthermore, in Ref. [31] multiplicative noise is directly implemented in the evolution equation for the spatial coordinate, yielding a very different physical scenario.

In conclusion, the model formalized by system (2) actually gives us the implementation of Lévy flights at the "microscopic" level admitting the notion of continuous trajectories. Indeed, fixing any small duration δt of the Lévy walker steps we can choose the time scale τ of model (2) such that $\delta t \gg \tau$ and, as a result, receive the Lévy statistics for the corresponding spatial steps. Moreover, expressions (14) and (15) demonstrate the equivalence of all the systems in asymptotic behavior for which the parameters v_a and τ are related by the expression $\sigma := v_a^\alpha \tau^{\alpha-1}$. In some sense, all the details of the microscopic implementation of Lévy flights are aggregated in two constants: the exponent α and the superdiffusion coefficient σ. In particular, the characteristic scale of the walker displacement during time t is $\bar{x}(t) \sim (\sigma t)^{1/\alpha}$.

Our approach has several immediate consequences. First of all, it yields an easily implementable procedure for the numerical simulation of Lévy processes based on the simulation of the Langevin equations (2). Second, it seems to be possible to attack the yet unsolved problem of the formulation of accurate boundary conditions for the generalized Fokker-Planck equations describing Lévy processes in finite domains and heterogeneous media. The crucial point of our treatment is the existence of quantities varying on three widely separated time scales $\widetilde{\delta t} \ll \tau \ll \delta t$. On time scales $\widetilde{\delta t}$ the Langevin equation is updated. In the well-defined limit of small $\widetilde{\delta t}$ the trajectory can be constructed with arbitrary precision. Furthermore, τ is connected with the relaxation time of the variable v and sets the overall time scale of the model. Finally, for δt the variation of the position x is fully Markovian and the systems behave according to the standard Lévy flight scenario. A similar approach is the treatment of the Kramers-Fokker-Planck equation describing diffusion of particles, which is obtained from Eq. (7) for the case of purely additive noise $g = \text{const}$. The so-called Smoluchowski limit $\tau \to 0$ leads to Einstein's diffusion equation. For equilibrium systems the fluctuation dissipation theorem relates linear damping and purely additive noise. The emergence of Lévy flights, however, is related to the presence of multiplicative noise, and, in turn, with nonequilibrium situations.

We note once more that for the analyzed random walks long movements are associated with fast speeds while short movements are associated with slow speeds. In this connection it is worth mentioning that such behavior is of considerable current interest in optimal random search theory and in the analysis of animal movement patterns (see, e.g., [32,33]). In these cases the searching phases tend to be associated with slow speeds while relocation phases tend to be associated with high speeds [2].

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